Interplanetary Trajectories in the Restricted Three-Body Problem

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The purpose of this paper is to present an approximate analytic solution to the restricted three-body problem valid for those initial conditions typical of interplanetary trajectories. The planar motion of a particle of negligible mass acted on by the gravitational attractions of two point masses m_0 and m_1 as the particle moves in the plane of motion of m_0 and m_1 to a small neighborhood of m_1 is studied in the framework of the restricted three-body problem for $\mu = m_1/m_0$ much less than one. A boundary-layer type of analysis is applied in order to obtain an approximate analytic solution to this problem. This type of analysis was first applied to this problem by P. A. Lagerstrom and J. Kevorkian. They first considered the two fixed centers problem and then the restricted three-body problem for a special class of initial conditions typical of certain Earth to moon trajectories, namely, those for which the initial angular momentum of the particle with respect to m_0 is $0(\mu^{1/2})$. This paper treats the problem using different independent variables in order to obtain a solution valid for a class of initial conditions typical of interplanetary trajectories; that is, those trajectories for which the initial angular momentum of the particle with respect to m_0 is 0(1).

Nomenclature

G	= universal gravitational constant
m_0	= mass of the largest body
m_1	= mass of the secondary body
m_2	= mass of the particle
μ	= m_1/m_0 = mass ratio
$\mathbf{r}$	= vector from m_0 to m_2
$\mathbf{r}_1$	= vector from m_0 to m_1
$\mathbf{r}_2$	= vector from m_1 to m_2
r	= $\lvert\mathbf{r}\rvert$ = magnitude of $\mathbf{r}$
β	= angle between $\mathbf{r}_1$ and $\mathbf{r}$
θ	= central angle of m_2 in m_0-centered nonrotating coordinates
ψ	= angle between $\mathbf{r}_1$ and $\mathbf{r}_2$
t	= time = central angle of m_1 in m_0-centered coordinates
t_0	= initial time
u	= $1/r$
h_0	= initial angular momentum of m_2 relative to m_0
e_0	= initial eccentricity of the instantaneous ellipse
a_0	= initial semimajor axis of the instantaneous ellipse
ω_0	= initial argument of pericenter of the instantaneous ellipse
θ_1	= zero-order angle of arrival
V_∞	= speed at infinity on the m_1-centered hyperbola
h_2	= angular momentum of the hyperbola
ω_2	= argument of pericenter of the hyperbola
e_2	= eccentricity of the hyperbola
θ_2	= central angle of m_2 in m_1-centered coordinates
Δ	= distance to the asymptote of the m_1-centered hyperbola
t_p	= time of pericenter passage on the m_1-centered hyperbola
(ξ, η)	= m_1-centered nonrotating coordinates
$(\dot{\ })$	= derivative of quantity with respect to time

IN this analysis it will be assumed that m_1 is in a circular orbit about m_0, although this is not necessary for the success of the method. The equations of motion of the particle with respect to m_0 are then given by

$$\ddot{\mathbf{r}} = -(\mathbf{r}/r^3) - \mu[\mathbf{r}_2/r_2^3 + \mathbf{r}_1] \tag{1}$$

where

$$\mathbf{r}_2 = \mathbf{r} - \mathbf{r}_1$$

and where the coordinates have been normalized by dividing distances by the constant m_0 to m_1 distance d and by dividing time by the constant $(d^3/Gm_0)^{1/2}$. As was mentioned in the abstract, it is assumed that the mass ratio $\mu = m_1/m_0$ is much less than one. The foregoing equations have the form of a singular perturbation problem. The perturbation term has a singularity when $r_2(t) = 0$. The problem will be treated as two ordinary perturbation problems: 1) when the particle is outside a small neighborhood of m_1, i. e., when $r_2(t) \geqq 0(\mu^{1/2})$; and 2) when the particle is inside a small neighborhood of m_1, i. e., when $r_2(t) \leqq 0(\mu^{1/2})$. The asymptotic expansions of the two perturbation solutions thus obtained are then matched in the boundary layer common to these two regions, i. e., when $r_2(t) = 0(\mu^{1/2})$. The perturbation solution for the particle outside a small neighborhood of m_1 is developed in a m_0-centered nonrotating coordinate system taking the true anomoly θ as the independent variable and effecting the matching in an m_1-centered nonrotating coordinate system with one axis parallel to the hyperbolic excess velocity. The analysis, therefore, differs considerably from the work of Lagerstrom and Kevorkian [1-3] in which the perturbation solution and matching can be carried out in a single rotating coordinate system using the distance along the m_0 m_1 line of centers as the independent variable. The basic ideas and results are, however, the same.

The perturbation solution for the particle outside a small neighborhood of m_1 and its asymptotic expansion will first be derived. The foregoing equations of motion (1) can be written in component form noting that the vectors

$$\mathbf{r}_1 = \mathbf{e}_r \cos\beta - \mathbf{e}_\theta \sin\beta$$

$$\mathbf{r}_2 = r\mathbf{e}_r - \mathbf{r}_1$$

where $\beta = \theta - t$ is the angle between $\mathbf{r}_1$ and $\mathbf{r}$, and θ is measured in a m_0-centered nonrotating reference frame (Fig. 1). The differential equations become

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{r^2} - \mu\left(\cos\beta + \frac{r - \cos\beta}{r_2^3}\right)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \mu\left(\sin\beta - \frac{\sin\beta}{r_2^3}\right) \tag{2}$$

The second equation implies that the angular momentum is a constant h_0, the initial value, plus a perturbation term

$$r^2\dot{\theta} = h_0 + \mu\int_{t_0}^{t} r\left(\sin\beta - \frac{\sin\beta}{r_2^3}\right)dt \tag{3}$$

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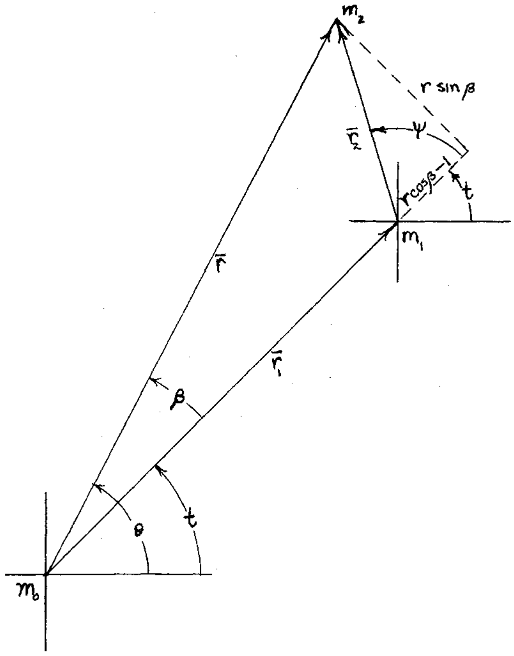

Fig. 1 Coordinate systems.

This equation can be used to write (2) in terms of θ as the independent variable. The dependent variables will be taken as t and $u = 1/r$. The foregoing equations become

$$\left.\begin{aligned} \frac{d^2u}{d\theta^2} + u &= \frac{1}{h_0^2} + \frac{\mu}{h_0^2}\left\{r^2\left(\cos\beta + \frac{r - \cos\beta}{r_2^3}\right) + r\frac{dr}{d\theta}\left(\sin\beta - \frac{\sin\beta}{r_2^3}\right) - \frac{2}{h_0^2}\int_{\theta_0}^{\theta} r^3 \times \left(\sin\beta - \frac{\sin\beta}{r_2^3}\right) d\theta\right\} + 0(\mu^2) \\ \frac{dt}{d\theta} &= \frac{r^2}{h_0} - \mu\frac{r^2}{h_0^3}\int_{\theta_0}^{\theta} r^3\left(\sin\beta - \frac{\sin\beta}{r_2^3}\right) d\theta + 0(\mu^2) \end{aligned}\right\} \quad (4)$$

where Eq. (3) has been used to write the integrals with respect to t as integrals with respect to θ to $0(\mu)$.

The solution to these equations can be written as an expansion in the small parameter μ:

$$u(\theta) = u_0(\theta) + \mu u_1(\theta) + \mu^2 u_2(\theta) + \ldots$$

$$t(\theta) = t_0(\theta) + \mu t_1(\theta) + \mu^2 t_2(\theta) + \ldots$$

The well-known zero-order solution to these equations for $e_0 < 1$ is (Ref. 4, p. 90)

$$r_0(\theta) = \frac{1}{u_0(\theta)} = \frac{h_0^2}{1 + e_0\cos(\theta - \omega_0)}$$

$$t_0(\theta) = t_0 + a_0^{3/2}\left[\sin^{-1}\left(\frac{(1 - e_0^2)^{1/2}\sin(\theta - \omega_0)}{1 + e_0\cos(\theta - \omega_0)}\right) - \frac{e_0(1 - e_0^2)^{1/2}\sin(\theta - \omega_0)}{1 + e_0\cos(\theta - \omega_0)}\right]_{\theta_0}^{\theta}$$

Similar equations hold for $e_0 \geqq 1$ (Ref. 4, p. 91). Define

$$\beta_0(\theta) = \theta - t_0(\theta)$$

$$r_{2_0}(\theta) = [1 - 2r_0(\theta)\cos\beta_0(\theta) + r_0^2(\theta)]^{1/2}$$

Then substituting the expansions for $u(\theta)$ and $t(\theta)$ into the differential equations (4) and equating the coefficients of like powers of μ determines the linear differential equations for $u_1(\theta)$ and $t_1(\theta)$, i. e.,

$$\left.\begin{aligned} \frac{d^2u_1}{d\theta^2} + u_1 &= \frac{1}{h_0^2}\left[r_0^2(\theta)\left(\cos\beta_0(\theta) + \frac{r_0(\theta) - \cos\beta_0(\theta)}{r_{2_0}^3(\theta)}\right) + r_0(\theta)r_0'(\theta)\left(\sin\beta_0(\theta) - \frac{\sin\beta_0(\theta)}{r_{2_0}^3(\theta)}\right) - \frac{2}{h_0^2}\int_{\theta_0}^{\theta} r_0^3(\theta)\left(\sin\beta_0(\theta) - \frac{\sin\beta_0(\theta)}{r_{2_0}^3(\theta)}\right) d\theta\right] \\ \frac{dt_1}{d\theta} &= -\frac{r_0^2(\theta)}{h_0^3}\int_{\theta_0}^{\theta} r_0^3(\theta)\left(\sin\beta_0(\theta) - \frac{\sin\beta_0(\theta)}{r_{2_0}^3(\theta)}\right) d\theta - \frac{2}{h_0} r_0^3(\theta)u_1(\theta) \end{aligned}\right\} \quad (5)$$

The initial conditions are chosen so that the particle reaches a small neighborhood of m_1. In fact, they are specified such that the unperturbed trajectory intersects m_1 at $\theta = \theta_1$, i. e., $r_0(\theta_1) = 1$ and $t_0(\theta_1) = \theta_1$. Small variations in these initial conditions can then be studied by standard error propagation techniques.

In order to determine the singular behavior of the first-order solution, i. e., of $u_1(\theta)$ and $t_1(\theta)$, as θ approaches θ_1, the singular terms appearing in (5) will be expanded about $\theta = \theta_1$. This is accomplished using the following expansions that are used throughout the paper:

$$\left.\begin{aligned} r_0(\theta) &= 1 - a\left(\frac{1 - h_0}{h_0}\right)(\theta_1 - \theta) + (a^2 - b) \times \left(\frac{1 - h_0}{h_0}\right)^2(\theta_1 - \theta)^2 + 0(\theta_1 - \theta)^3 \\ t_0(\theta) &= \theta_1 - \frac{1}{h_0}(\theta_1 - \theta) + \frac{a}{h_0^2}(1 - h_0)(\theta_1 - \theta)^2 + 0(\theta_1 - \theta)^3 \\ \beta_0(\theta) &= \left(\frac{1 - h_0}{h_0}\right)(\theta_1 - \theta)\left[1 - \frac{a}{h_0}(\theta_1 - \theta)\right] + 0(\theta_1 - \theta)^3 \end{aligned}\right\} \quad (6)$$

where

$$a = \left[\frac{e_0\sin(\theta_1 - \omega_0)}{1 + e_0\cos(\theta_1 - \omega_0)}\right]\left(\frac{h_0}{1 - h_0}\right) = \frac{e_0\sin(\theta_1 - \omega_0)}{h_0(1 - h_0)}$$

$$b = -\frac{1}{2}\left[\frac{e_0\cos(\theta_1 - \omega_0)}{1 + e_0\cos(\theta_1 - \omega_0)}\right]\left(\frac{h_0}{1 - h_0}\right)^2 = -\frac{e_0\cos(\theta_1 - \omega_0)}{2(1 - h_0)^2}$$

since

$$1 = r_0(\theta_1) = \frac{h_0^2}{1 + e_0\cos(\theta_1 - \omega_0)}$$

The other constants appearing in these equations are defined in the Nomenclature. Using these expansions, the behavior of the singular terms in (5) can be described

$$\frac{\sin\beta_0(\theta)}{r_{2_0}^3(\theta)} = \frac{K_0}{(\theta_1 - \theta)^2} + \frac{K_1}{(\theta_1 - \theta)} + \Psi(\theta)$$

$$\frac{r_0(\theta) - \cos\beta_0(\theta)}{r_{2_0}^3(\theta)} = \frac{-aK_0}{(\theta_1 - \theta)^2} + \frac{K_2}{(\theta_1 - \theta)} + \Phi(\theta)$$

where the constants

$$K_0 = \frac{h_0^2(1 - h_0)}{(1 + a^2)^{3/2}|1 - h_0|^3}$$

$$K_1 = \frac{aK_0}{h_0}(2 - 3h_0)$$

$$K_2 = -K_0\left[1 + \left(\frac{1-h_0}{h_0}\right) 2a^2\right]$$

and the functions

$$\Psi(\theta) = \frac{\sin\beta_0(\theta)}{r_{2_0}{}^3(\theta)} - \frac{K_0}{(\theta_1 - \theta)^2} - \frac{K_1}{(\theta_1 - \theta)}$$

$$\Phi(\theta) = \frac{r_0(\theta) - \cos\beta_0(\theta)}{r_{2_0}{}^3(\theta)} + \frac{aK_0}{(\theta_1 - \theta)^2} - \frac{K_2}{(\theta_1 - \theta)}$$

remain bounded as θ approaches θ_1. The behavior of the singular and bounded parts of $\sin\beta_0(\theta)/r_{2_0}{}^3(\theta)$ for $\theta_1 = \pi$ is shown graphically in Fig. 2.

The integrals necessary to solve the system (5) will now be written as a singular part plus the integral of a bounded function. Only elementary operations and the concept of adding and subtracting the singular part of a function in order to write it as an integrable function plus a bounded function were used in deriving these integrals:

$$\int_{\theta_0}^{\theta} r_0{}^3(\theta)\left[\sin\beta_0(\theta) - \frac{\sin\beta_0(\theta)}{r_{2_0}{}^3(\theta)}\right] d\theta = \frac{-K_0}{(\theta_1 - \theta)} - \frac{aK_0}{h_0}\ln(\theta_1 - \theta) + G_1(\theta)$$

where the function

$$G_1(\theta) = \frac{K_0}{(\theta_1 - \theta_0)} + \frac{aK_0}{h_0}\ln(\theta_1 - \theta_0) + \int_{\theta_0}^{\theta}\Psi_1(\theta)d\theta$$

$$\Psi_1(\theta) = -r_0{}^3(\theta)[\Psi(\theta) + \sin\beta_0(\theta)] + K_0\left[\frac{1 - r_0{}^3(\theta) - 3a[(1-h_0)/h_0](\theta_1 - \theta)}{(\theta_1 - \theta)^2}\right] + K_1\left[\frac{1 - r_0{}^3(\theta)}{(\theta_1 - \theta)}\right]$$

is bounded as θ approaches θ_1.
Using vector notation to minimize the writing,

$$\int_{\theta_0}^{\theta} r_0(\theta)r_0'(\theta)\left[\sin\beta_0(\theta) - \frac{\sin\beta_0(\theta)}{r_{2_0}{}^3(\theta)}\right] \times \begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} d\theta = \int_{\theta_0}^{\theta}\Psi_2(\theta)\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} d\theta - \frac{aK_0(1-h_0)}{h_0(\theta_1 - \theta)}\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} + \frac{aK_0(1-h_0)}{h_0(\theta_1 - \theta_0)}\begin{pmatrix}1\\ 0\end{pmatrix} - M_0\begin{pmatrix}Ci(\theta_1 - \theta) - Ci(\theta_1 - \theta_0)\\ Si(\theta_1 - \theta) - Si(\theta_1 - \theta_0)\end{pmatrix}$$

where

$$Ci(x) = -\int_x^{\infty}\frac{\cos x'}{x'}dx' \qquad Si(x) = \int_0^x \frac{\sin x'}{x'}dx'$$

are the cosine-integral and sine-integral functions, respectively. As x approaches zero, $Ci(x)$ behaves like $\ln(x) + \gamma$, where γ is Euler's constant and $Si(x)$ approaches zero. The function

$$\Psi_2(\theta) = r_0(\theta)r_0'(\theta)[\sin\beta_0(\theta) - \Psi(\theta)] - K_1\left[\frac{r_0(\theta)r_0'(\theta) - a[(1-h_0)/h_0]}{(\theta_1 - \theta)}\right] - K_0\left[\frac{[r_0(\theta)r_0'(\theta) - a[(1-h_0)/h_0] - (2b - 3a^2)[(1 - h_0/h_0]^2(\theta_1 - \theta)}{(\theta_1 - \theta)^2}\right]$$

is bounded as θ approaches θ_1. Finally,

$$M_0 = aK_0\left(\frac{1-h_0}{h_0}\right)\begin{bmatrix}-\sin(\theta_1 - \theta_0) & \cos(\theta_1 - \theta_0)\\ \cos(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0)\end{bmatrix} - K_0\left(\frac{1-h_0}{h_0}\right)^2\left(2b - \frac{a^2}{1-h_0}\right)\begin{bmatrix}\cos(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0)\\ \sin(\theta_1 - \theta_0) & -\cos(\theta_1 - \theta_0)\end{bmatrix}$$

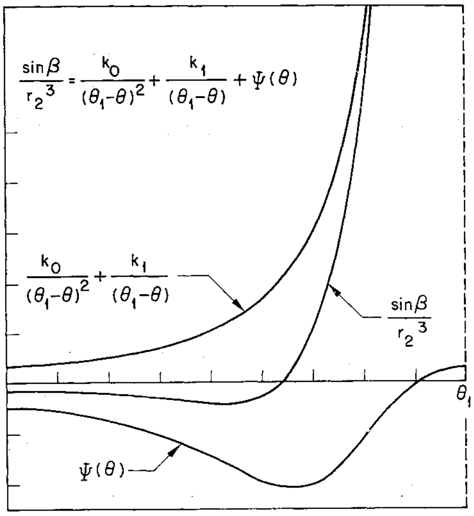

Fig. 2 Singular perturbation.

Similarly,

$$\int_{\theta_0}^{\theta} r_0{}^2(\theta)\left[\cos\beta_0(\theta) + \frac{r_0(\theta) - \cos\beta_0(\theta)}{r_{2_0}{}^3(\theta)}\right]\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} d\theta = \int_{\theta_0}^{\theta}\Phi_1(\theta)\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} d\theta - \frac{aK_0}{(\theta_1 - \theta)}\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} + \frac{aK_0}{(\theta_1 - \theta_0)}\begin{pmatrix}1\\ 0\end{pmatrix} - M_1\begin{pmatrix}Ci(\theta_1 - \theta) - Ci(\theta_1 - \theta_0)\\ Si(\theta_1 - \theta) - Si(\theta_1 - \theta_0)\end{pmatrix}$$

where the function

$$\Phi_1(\theta) = r_0{}^2(\theta)[\cos\beta_0(\theta) + \Phi(\theta)] + aK_0\left[\frac{1 - r_0{}^2(\theta) - 2a[(1-h_0)/h_0](\theta_1 - \theta)}{(\theta_1 - \theta)^2}\right] + K_2\left[\frac{r_0{}^2(\theta) - 1}{(\theta_1 - \theta)}\right]$$

is bounded as θ approaches θ_1; furthermore,

$$M_1 = aK_0\begin{bmatrix}-\sin(\theta_1 - \theta_0) & \cos(\theta_1 - \theta_0)\\ \cos(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0)\end{bmatrix} - K_0\begin{bmatrix}\cos(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0)\\ \sin(\theta_1 - \theta_0) & -\cos(\theta_1 - \theta_0)\end{bmatrix}$$

The last integral that is needed to solve the u_1 equation is

$$\int_{\theta_0}^{\theta}\int_{\theta_0}^{\theta} r_0{}^3(\theta')\left[\sin\beta_0(\theta') - \frac{\sin\beta_0(\theta')}{r_{2_0}{}^3(\theta')}\right]d\theta'\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} d\theta = \int_{\theta_0}^{\theta}\left\{G_1(\theta)\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} + \frac{aK_0}{h_0}[\ln(\theta_1 - \theta) - 1] \times (\theta_1 - \theta)\begin{pmatrix}\sin(\theta - \theta_0)\\ -\cos(\theta - \theta_0)\end{pmatrix}\right\} d\theta + \frac{aK_0}{h_0}(\theta_1 - \theta)[\ln(\theta_1 - \theta) - 1]\begin{pmatrix}\cos(\theta - \theta_0)\\ \sin(\theta - \theta_0)\end{pmatrix} - \frac{aK_0}{h_0}(\theta_1 - \theta_0)[\ln(\theta_1 - \theta_0) - 1]\begin{pmatrix}1\\ 0\end{pmatrix} + M_3\begin{pmatrix}Ci(\theta_1 - \theta) - Ci(\theta_1 - \theta_0)\\ Si(\theta_1 - \theta) - Si(\theta_1 - \theta_0)\end{pmatrix}$$

with the constant matrix

$$M_3 = K_0 \begin{bmatrix} \cos(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0) \\ \sin(\theta_1 - \theta_0) & -\cos(\theta_1 - \theta_0) \end{bmatrix}$$

These integrals together with the fact that the differential equation

$$(d^2u_1/d\theta^2) + u_1 = F(\theta)$$

with the initial conditions

$$u_1(\theta_0) = u_1'(\theta_0) = 0$$

has the solution

$$u_1(\theta) = \sin(\theta - \theta_0) \int_{\theta_0}^{\theta} F(\theta) \cos(\theta - \theta_0) d\theta - \cos(\theta - \theta_0) \int_{\theta_0}^{\theta} F(\theta) \sin(\theta - \theta_0) d\theta$$

determine the solution of the first equation in (5), i.e.,

$$u_1(\theta) = \sin(\theta - \theta_0)[c_0 + G_3(\theta)] - \cos(\theta - \theta_0) G_2(\theta) + \sin(\theta_1 - \theta)[-A_1 + A_0 Ci(\theta_1 - \theta) + B_0 Si(\theta_1 - \theta)] + \cos(\theta_1 - \theta)[B_1 + B_0 Ci(\theta_1 - \theta) - A_0 Si(\theta_1 - \theta)]$$

where the functions

$$\begin{pmatrix} G_2(\theta) \\ G_3(\theta) \end{pmatrix} = \frac{1}{h_0^2} \int_{\theta_0}^{\theta} \left\{ \left[\Phi_1(\theta) + \Psi_2(\theta) - \frac{2}{h_0^2} G_1(\theta) \right] \times \begin{pmatrix} \sin(\theta - \theta_0) \\ \cos(\theta - \theta_0) \end{pmatrix} + \frac{2aK_0}{h_0^5} (\theta_1 - \theta)[\ln(\theta_1 - \theta) - 1] \times \begin{pmatrix} \cos(\theta - \theta_0) \\ -\sin(\theta - \theta_0) \end{pmatrix} \right\} d\theta$$

and the constants

$$c_0 = \frac{aK_0}{h_0^3(\theta_1 - \theta_0)} + \frac{2aK_0}{h_0^5} (\theta_1 - \theta_0)[\ln(\theta_1 - \theta_0) - 1]$$

$$A_0 = (K_0/h_0^4)[1 + a^2(1 - h_0)]$$

$$B_0 = aK_0/h_0^3$$

$$A_1 = A_0 Ci(\theta_1 - \theta_0) + B_0 Si(\theta_1 - \theta_0)$$

$$B_1 = A_0 Si(\theta_1 - \theta_0) - B_0 Ci(\theta_1 - \theta_0)$$

The important thing to note in this equation for $u_1(\theta)$ is that the $1/(\theta_1 - \theta)$ terms have canceled one another and that the $Ci(\theta_1 - \theta)$ terms, which behave like $\ln(\theta_1 - \theta) + \gamma$ as θ approaches θ_1, therefore determine the singular behavior of $u_1(\theta)$. The first-order solution for the radial distance as a function of θ can then be written as

$$r(\theta) = r_0(\theta) - \mu r_0^2(\theta) u_1(\theta) \tag{7}$$

The equation for $t_1(\theta)$ from (5) can be written as

$$t_1(\theta) = \frac{-2}{h_0} \int_{\theta_0}^{\theta} r_0^3(\theta) u_1(\theta) d\theta + \frac{1}{h_0^3} \int_{\theta_0}^{\theta} r_0^2(\theta) \left[\frac{K_0}{(\theta_1 - \theta)} + \frac{aK_0}{h_0} \ln(\theta_1 - \theta) - G_1(\theta) \right] d\theta$$

Integrating out the singular parts of these functions determines the first-order solution for the time as a function of θ:

$$t(\theta) = t_0(\theta) - \mu \left\{ \frac{K_0}{h_0^3} \ln(\theta_1 - \theta) - G_4(\theta) - \frac{aK_0}{h_0^4} (\theta_1 - \theta)[\ln(\theta_1 - \theta) - 1] \right\} \tag{8}$$

where the function

$$G_4(\theta) = c_1 + \int_{\theta_0}^{\theta} \Phi_2(\theta) d\theta$$

$$\Phi_2(\theta) = \frac{1}{h_0^3} \left\{ \frac{K_0[r_0^2(\theta) - 1]}{(\theta_1 - \theta)} + \frac{aK_0}{h_0} [r_0^2(\theta) - 1] \ln(\theta_1 - \theta) - r_0^2(\theta) G_1(\theta) \right\}$$

is bounded as θ approaches θ_1; furthermore

$$c_1 = \frac{K_0}{h_0^3} \ln(\theta_1 - \theta_0) - \frac{aK_0}{h_0^4} [\ln(\theta_1 - \theta_0) - 1](\theta_1 - \theta_0)$$

It will be assumed in these equations that $h_0 \gg \mu^{1/2}$ and that $|1 - h_0| \gg \mu^{1/2}$.

It can be shown that the dominant second-order terms have the form $\mu^2 \ln(\theta_1 - \theta)/(\theta_1 - \theta)$, and that the perturbation solution and the asymptotic expansions to follow are correct to $0(\mu^{3/2})$ for $\theta_1 - \theta = 0(\mu^{1/2})$.

The asymptotic expansions of Eqs. (7) and (8) describing the m_0-centered perturbed conic will now be determined. Let $\theta_1 - \theta = \mu^{1/2}\varphi$ define the angle φ. Then the following asymptotic behavior of the perturbed conic for $\varphi = 0(1)$ is determined by expanding Eqs. (7) and (8) about $\theta = \theta_1$:

$$\left.\begin{aligned} r(\theta) &= 1 - \mu^{1/2} a \left(\frac{1 - h_0}{h_0} \right) \varphi + \mu(a^2 - b) \times \left(\frac{1 - h_0}{h_0} \right)^2 \varphi^2 - \mu[B_0 \ln(\mu^{1/2}\varphi) + G_5(\theta_1)] + 0(\mu^{3/2}) \\ t(\theta) &= \theta_1 - \frac{\mu^{1/2}\varphi}{h_0} + \mu \frac{a(1 - h_0)}{h_0^2} \varphi^2 - \mu \left[\frac{K_0}{h_0^3} \ln(\mu^{1/2}\varphi) - G_4(\theta_1) \right] + 0(\mu^{3/2}) \\ \beta(\theta) &= \mu^{1/2} \left(\frac{1 - h_0}{h_0} \right) \varphi - \mu \frac{a(1 - h_0)}{h_0^2} \varphi^2 + \mu \left[\frac{K_0}{h_0^3} \ln(\mu^{1/2}\varphi) - G_4(\theta_1) \right] + 0(\mu^{3/2}) \end{aligned}\right\} \tag{9}$$

where the constant

$$G_5(\theta_1) = \sin(\theta_1 - \theta_0)[c_0 + G_3(\theta_1)] - \cos(\theta_1 - \theta_0) G_2(\theta_1) + B_1 + \gamma B_0$$

The fact that

$$Ci(x) = \gamma + \ln(x) - \frac{x^2}{2 \cdot 2!} + \ldots$$

and that

$$Si(x) = x - \frac{x^3}{3 \cdot 3!} + \ldots$$

has been used in carrying out these expansions.

These equations are next related to a suitably oriented m_1-centered, nonrotating coordinate system in which the matching is easily performed.† The angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ (Fig. 1) is

$$\psi = \tan^{-1}\left(\frac{r \sin\beta}{r \cos\beta - 1} \right)$$

The angle $\varphi_2 = \psi + t - \theta_1$ is then the central angle of $\mathbf{r}_2$ measured in a nonrotating, m_1-centered coordinate system. If this system is rotated through the constant angle $\alpha = \tan^{-1}(1/a)$, chosen in the first quadrant for $h_0 \leq 1$ and in the third quadrant for $h_0 > 1$, the components of $\mathbf{r}_2$ in the resulting m_1-centered nonrotating coordinate system will be

$$\xi = r_2 \cos(\varphi_2 + \alpha) = \pm r_2 \left(\frac{\sin\varphi_2 - a \cos\varphi_2}{(1 + a^2)^{1/2}} \right)$$

$$\eta = r_2 \sin(\varphi_2 + \alpha) = \pm r_2 \left(\frac{a \sin\varphi_2 + \cos\varphi_2}{(1 + a^2)^{1/2}} \right)$$

the plus sign corresponding to $h_0 \leq 1$ and the minus sign to $h_0 > 1$. From the definition of φ_2 and the fact that by Eq. (9)

$$t - \theta_1 = -\frac{\mu^{1/2}\varphi}{h_0} + 0(\mu)$$

$$r_2 = [1 - r \cos\beta + r^2]^{1/2} = 0(\mu^{1/2})$$

† This approach is due to J. V. Breakwell.

for $\varphi \leqq 0(1)$, it follows that

$$r_2 \sin\varphi_2 = r_2 \sin\psi - \frac{\mu^{1/2}\varphi}{h_0} r_2 \cos\psi + 0(\mu^{3/2})$$

$$r_2 \cos\varphi_2 = r_2 \cos\psi + \frac{\mu^{1/2}\varphi}{h_0} r_2 \sin\psi + 0(\mu^{3/2})$$

for $\varphi \leqq 0(1)$. Finally, since (see Fig. 1)

$$r_2 \sin\psi = r \sin\beta \qquad r_2 \cos\psi = r \cos\beta - 1$$

and since the asymptotic behavior of r and β is given by Eq. (9), the asymptotic behavior of ξ and η can be determined using the foregoing equations, i. e.,

$$\left.\begin{aligned} \xi &= \mu^{1/2} \frac{|1 - h_0|(1 + a^2)^{1/2}}{h_0} \varphi + \\ &\mu\left[\frac{|K_0|}{h_0^3}(1 + a^2)^{1/2} \ln(\mu^{1/2}\varphi) + \frac{aG_5(\theta_1) - G_4(\theta_1)}{(1 + a^2)^{1/2}} - \right. \\ &\left. a(1 + a^2)^{1/2}\left(\frac{1 - h_0}{h_0}\right)^2 \varphi^2\right] + 0(\mu^{3/2}) \\ \eta &= \frac{\mp\mu[aG_4(\theta_1) + G_5(\theta_1)]}{(1 + a^2)^{1/2}} + 0(\mu^{3/2}) \end{aligned}\right\} \quad (10)$$

To see that the φ^2 terms drop out in the η equation, it is necessary to use the fact that $b = 1/(1 - h_0) - \frac{1}{2}$, which follows from the definition of the constant b.

As might be expected, these equations together with the time equation in (9) will be shown to be the same as the asymptotic expansions of a m_1-centered hyperbola with the asymptote of the hyperbola parallel to the ξ axis. To facilitate this matching, the first equation in (10) is used to eliminate φ in the t and η equations in (9) and (10), respectively. For instance, t and η are written as functions of ξ for $\varphi \leqq 0(1)$ or $\xi \leqq 0(\mu^{1/2})$ to order $\mu^{3/2}$:

$$\left.\begin{aligned} t &= t_0(\theta_1) - \frac{\xi}{|1 - h_0|(1 + a^2)^{1/2}} + \mu\left[\frac{|K_0|}{h_0^2|1 - h_0|}\times\right. \\ &\left.\ln\left(\frac{h_0\xi}{|1 - h_0|(1 + a^2)^{1/2}}\right) + G_6(\theta_1)\right] + 0(\mu^{3/2}) \\ \eta &= \frac{\pm\mu G_7(\theta_1)}{(1 + a^2)^{1/2}} + 0(\mu^{3/2}) \end{aligned}\right\} \quad (11)$$

Note that from the definition of K_0

$$\frac{|K_0|}{h_0^2|1 - h_0|} = \frac{1}{(1 + a^2)^{3/2}|1 - h_0|^3}$$

in these equations. And the constants

$$G_6(\theta_1) = \frac{aG_5(\theta_1) - G_4(\theta_1)}{(1 + a^2)|1 - h_0|} + G_4(\theta_1)$$

$$G_7(\theta_1) = -[aG_4(\theta_1) + G_5(\theta_1)]$$

are definite integrals of bounded functions from θ_0 to θ_1.

Next the perturbation solution for the particle inside a small neighborhood of m_1 and its asymptotic expansion will be derived. The differential equation of motion (1) can be written as

$$\ddot{\mathbf{r}}_2 = -\mu \frac{\mathbf{r}_2}{r_2^3} - \left(\frac{\mathbf{r}}{r^3} - \mathbf{r}_1\right)$$

since $\mathbf{r}_2 = \mathbf{r} - \mathbf{r}_1$ and $\ddot{\mathbf{r}}_1 = (1 + \mu)\mathbf{r}_1$. The unperturbed solution satisfying the equation

$$\ddot{\mathbf{r}}_2 = -\mu(\mathbf{r}_2/r_2^3)$$

has the form of a hyperbolic motion (for positive energy), which can be described (Ref. 5, p. 178) in terms of the variable

$$F = 2\tanh^{-1}\left[\left(\frac{e_2 - 1}{e_2 + 1}\right)^{1/2} \tan\left(\frac{\theta_2 - \omega_2}{2}\right)\right] \quad (12a)$$

by the equations

$$\left.\begin{aligned} t_p - t &= \frac{\mu h_2^3}{(e_2^2 - 1)^{3/2}} (e_2 \sinh|F| - |F|) \\ r_2 &= \frac{\mu h_2^2}{(e_2^2 - 1)} (e_2 \cosh F - 1) \qquad F \leq 0 \end{aligned}\right\} \quad (12b)$$

The asymptotic behavior of these equations for $r_2 \geqq 0(\mu^{1/2})$ will now be determined. This amounts to determining the behavior far out on the hyperbola [since $\mu^{1/2} \gg \mu$ and the pericenter distance is $0(\mu)$]. Let the auxiliary variable $z = \mu e^{|F|}$. Equations (12) in terms of this variable become

$$\left.\begin{aligned} \theta_2 &= \omega_2 - 2 \tan^{-1}\left[\left(\frac{e_2 + 1}{e_2 - 1}\right)^{1/2}\left(\frac{z - \mu}{z + \mu}\right)\right] \\ t_p - t &= \frac{\mu h_2^3}{(e_2^2 - 1)^{3/2}} \left[\frac{e_2}{2}\left(\frac{z}{\mu} - \frac{\mu}{z}\right) - \ln\frac{z}{\mu}\right] \\ r_2 &= \frac{\mu h_2^2}{(e_2^2 - 1)} \left[\frac{e_2}{2}\left(\frac{z}{\mu} + \frac{\mu}{z}\right) - 1\right] \end{aligned}\right\} \quad (13)$$

Choosing the argument of pericenter as

$$\omega_2 = 2 \tan^{-1}\left[\left(\frac{e_2 + 1}{e_2 - 1}\right)^{1/2}\right]$$

and measuring θ_2 from the ξ axis orients the hyperbola with its asymptote parallel to the ξ axis. This follows since, with this choice of ω_2,

$$\theta_2 = 2 \tan^{-1}\left[\frac{\mu(e_2^2 - 1)^{1/2}}{e_2 z - \mu}\right]$$

and θ_2 approaches zero as z increases without bound, i. e., as r_2 increases without bound. The asymptotic expansions describing the behavior of the hyperbola for $z \geqq 0(\mu^{1/2})$, i.e., for $r_2 \geqq 0(\mu^{1/2})$, can then be written in component form as

$$\left.\begin{aligned} \xi &= r_2 \cos\theta_2 = \frac{\mu h_2^2}{e_2^2 - 1}\left(\frac{e_2 z}{2\mu} - 1\right) + 0\left(\frac{\mu^2}{z}\right) \\ \eta &= r_2 \sin\theta_2 = \frac{\mu h_2^2}{(e_2^2 - 1)^{1/2}} + 0\left(\frac{\mu^2}{z}\right) \\ t_p - t &= \frac{\mu h_2^3}{(e_2^2 - 1)^{3/2}}\left(\frac{e_2 z}{2\mu} - \ln\frac{z}{\mu}\right) + 0\left(\frac{\mu^2}{z}\right) \end{aligned}\right\} \quad (14)$$

Then using the first equation in the foregoing to eliminate the auxiliary variable z, these equations can be written in terms of ξ as the independent variable for $\xi \geqq 0(\mu^{1/2})$:

$$\left.\begin{aligned} t_p - t &= \frac{\xi}{V_\infty} + \frac{\mu}{V_\infty^3}\left[1 - \ln\left(\frac{2V_\infty^2\xi}{\mu e_2}\right)\right] + 0(\mu^{3/2}) \\ \eta &= \frac{\mu h_2}{V_\infty} + 0(\mu^{3/2}) \end{aligned}\right\} \quad (15)$$

where

$$V_\infty = (e_2^2 - 1)^{1/2}/h_2$$

It can also be shown that the perturbation terms due to m_0 are $0(\mu^{3/2})$ for $r_2 \leqq 0(\mu^{1/2})$. This then completes the description of the asymptotic behavior of the m_1-centered hyperbola.

The matching is now performed, i.e., the constants of the m_1-centered hyperbola are determined by comparing the asymptotic expansions of the perturbed conic about m_0 as given by (11) and the hyperbola about m_1 as given by (15). The two expansions are identical provided the constants of the m_1-centered hyperbola are chosen as

$$V_\infty = |1 - h_0|(1 + a^2)^{1/2}$$

$$h_2 = (1 - h_0)G_7(\theta_1)$$

$$t_p = t_0(\theta_1) + \frac{\mu}{V_\infty^3}\left[1 + \ln\left(\frac{\mu h_0 e_2}{2V_\infty^3}\right)\right] + \mu G_6(\theta_1)$$

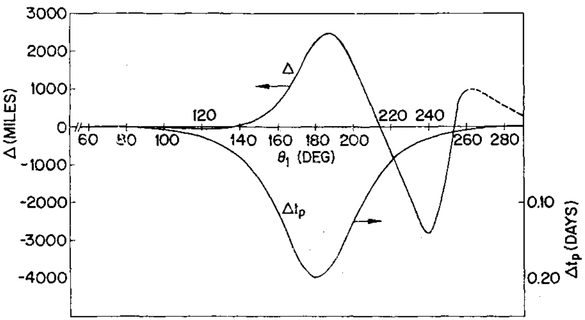

Fig. 3 Effect of Mar's perturbation on certain Earth-Mars trajectories.

where

$$e_2 = (1 + V_\infty{}^2 h_2{}^2)^{1/2}$$

Recall that the orientation of the m_1-centered coordinates is determined by a rotation through the angle

$$\alpha = \tan^{-1}(1/a)$$

chosen in the first quadrant if $h_0 \leqq 1$ and in the third quadrant if $h_0 > 1$. This then completes the matching and specifies the constants of the m_1-centered hyperbola.

Since the inner and outer solutions are written in terms of different independent variables that are not related in any elementary fashion, a composite solution cannot be obtained. However, a uniformly valid asymptotic approximation correct to $0(\mu^{3/2})$ is given by Eqs. (7, 8, and 13), provided they are restricted to the appropriate regions. This follows from the fact that their asymptotic expansions (10) [with the time equation given in (9)] and (14), which are correct to $0(\mu^{3/2})$ for $\varphi \leqq 0(1)$ and $z \geqq 0(\mu^{1/2})$, match [when they are both written in terms of ξ for $\xi = 0(\mu^{1/2})$ i.e., $\varphi = 0(1)$, i.e., $z = 0(\mu^{1/2})$] in the boundary layer common to the two regions. Moreover, once the constants of the m_1-centered hyperbola have been determined in terms of the given initial conditions and certain definite integrals of bounded functions, the effect of m_1 on trajectories that reach a small neighborhood of m_1 can be studied. For example, the perturbation by Mars on a class of Earth-Mars trajectories leaving a massless Earth at perihelion, with initial conditions such that the unperturbed heliocentric conic intersects Mars, was studied. The definite integrals were evaluated numerically, and the results are plotted as a function of transfer angle θ_1 in Fig. 3. It can be seen that the perturbation by Mars causes the particle to arrive earlier by an amount $\Delta t_p = t_p - t_0(\theta_1)$ and causes the asymptote to be deflected by an amount

$$\Delta = \frac{\mu h_2{}^2}{(e_2{}^2 - 1)^{1/2}} = \frac{\mu |G_7(\theta_1)|}{(1 + a^2)^{1/2}}$$

the deflection being away from the sun for transfers less than approximately 214° and toward the sun for larger transfer angles.

In Fig. 3, corresponding to the condition $|1 - h_0| \gg \mu^{1/2}$, there is a small interval about $\theta_1 = \pi/2$ for which no points on the curves were obtained. The curves were continued smoothly across this interval of a few degrees to obtain the curves as shown in Fig. 3.

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